MATH 2050C Lecture on 3/27/2020

Note:
- no lectures / tutorials next week (Reading Week)
- arrangements for midterm (Apr 15) & final (May 5) on course webpage.
- PS8 due today, PS9 posted (due 2 weeks later)

Recall: "Sequential Criteria"

① prove divergence of limit of functions (choose a suitable seq. $(x_n) \to c$)

② carry over thm about limit of seq. to limit of function

Limit Thms for functions

IDEA: Thm. for limit of seq. $\xrightarrow{\text{seq. criteria}}$ Thm. for limit of fcn.

Thm:

(1) $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

(2) $\lim_{x \to c} (fg)(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

(3)* $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

provided that:
- $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ (same domain)
- $\lim_{x \to c} f(x)$, $\lim_{x \to c} g(x)$ exist

* · Further assume $\lim_{x \to c} g(x) \neq 0$ (and $g(x) \neq 0$ "near c")

Note: $f \pm g, \, fg : A \to \mathbb{R}$

$f/g : A \setminus \{x \mid g(x) = 0\} \to \mathbb{R}$

Pf: of (2) Two alternatives: (i) ε-δ def$^{\underline{n}}$ (Ex) ; OR (ii) Seq. criteria. here.

Take ANY seq. (x_n) in A s.t. $\lim (x_n) = c$ and $x_n \neq c \;\; \forall n \in \mathbb{N}$.

WANT: $\underbrace{\big((fg)(x_n)\big)}_{\big(f(x_n)\, g(x_n)\big)} \longrightarrow \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ (#)

Apply seq. criteria to f $\Rightarrow$ $(f(x_n)) \longrightarrow \lim_{x \to c} f(x)$

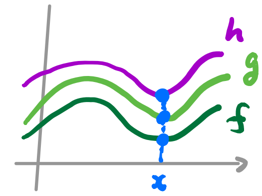
Apply seq. criteria to g $\Rightarrow$ $(g(x_n)) \longrightarrow \lim_{x \to c} g(x)$

$\Rightarrow$ (#) by Limit Thm for seq.

∎

Examples: $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ (if $c \neq 0$) ; $\lim_{x\to 2} \frac{x^3-4}{x+1} = \frac{4}{3}$

$$\lim_{x\to 2} \frac{x^2-4}{3x-6} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{3\cancel{(x-2)}} \overset{\text{(why?)}}{=} \lim_{x\to 2} \frac{x+2}{3} = \frac{4}{3}.$$



Squeeze Thm for functions

Let $f, g, h : A \to \mathbb{R}$ be s.t. "$f(x) \leq g(x) \leq h(x) \quad \forall x \in A, x \neq c$" (near c)

If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = L$

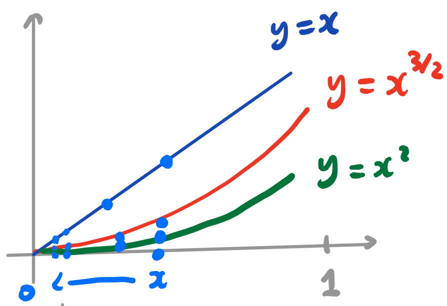
Pf: (Exercise!)

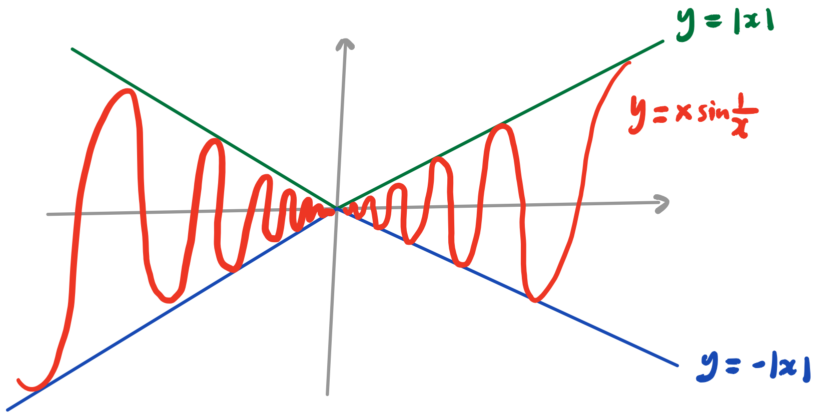
Note: This is NOT the same as

$f(x) \leq g(y) \leq h(z) \quad \forall x,y,z \in A$

Examples: $\lim_{x\to 0} x^{3/2} = 0$ $(x \geq 0)$ since $x^2 \leq x^{3/2} \leq x$ when $0 \leq x \leq 1$

$\lim_{x\to 0} x \sin\frac{1}{x} = 0$ $(x \neq 0)$ since $-|x| \leq x\sin\frac{1}{x} \leq |x| \quad \forall x \in \mathbb{R}, x \neq 0.$





Two useful facts about $\lim_{x\to c} f(x)$:

Prop: $\lim_{x\to c} f(x) = L \Rightarrow$ f is bdd in a neighborhood of c

i.e. $\exists M > 0$ and $\exists \delta > 0$ s.t.

$$|f(x)| \leq M \quad \forall x \in A \text{ s.t. } |x-c| < \delta$$

[Recall: For seq, $\lim (x_n)$ exists $\Rightarrow$ (x_n) bdd.]

Pf: Take $\varepsilon = 1$, by def$^{\underline{n}}$ of limit, $\exists \delta = \delta(1) > 0$ s.t.

$$|f(x) - L| < \varepsilon = 1 \quad \forall\ 0 < |x-c| < \delta$$

(Δ ineq.)
$\Rightarrow \quad |f(x)| \leq 1 + |L| \qquad \forall \; 0 < |x-c| < \delta$

$\Rightarrow \quad |f(x)| \leq M := \max\{1+|L|, |f(c)|\} \qquad \forall \; |x-c| < \delta$

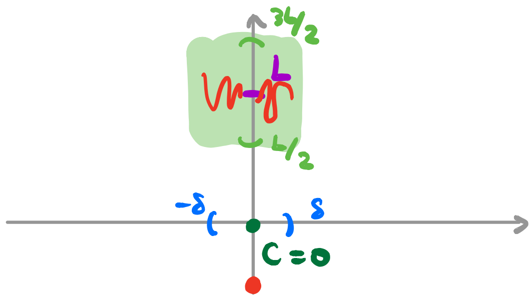
$\square$

Remark: $f: A \to \mathbb{R}$ may not be bdd globally. (e.g. $f(x) = x$)

Prop: $\lim_{x\to c} f(x) = L > 0 \Rightarrow \exists \delta > 0$ s.t. $f(x) > 0 \quad \forall \; 0 < |x-c| < \delta$.

Remark: False when $L = 0$.

Proof: Almost the same for seq. (take $\varepsilon = \frac{L}{2} > 0$ in def$^{\underline{n}}$ of limit)



Ex: Write it up!

$\square$

Summary: ① $\mathbb{R}$ as complete, ordered field

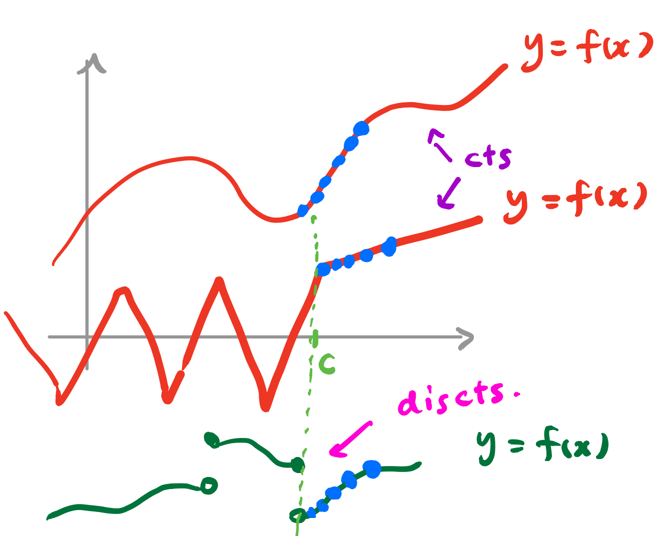
② $\lim (x_n)$

③ $\lim_{x\to c} f(x)$

④ "continuity"

Continuity (Ch. 5)

Q: When is a function $f: A \to \mathbb{R}$ "continuous" (at a point $c \in A$)?



Idea:

(1) "Continuity" means "drawing the graph $y=f(x)$ in one stroke".

(2) "cts at c" means

"$f(x) \approx f(c)$ when $x \approx c$"

$\Updownarrow$

$\lim_{x\to c} f(x) = f(c)$

Def$^{\underline{n}}$: Let $f: A \to \mathbb{R}$ be a function, and $c \in A$.

" f is continuous at c " iff $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \forall x \in A, \ |x - c| < \delta \qquad (*)$$

Remarks: (1) This is very similar to the def$^{\underline{n}}$ of limit of f at c

BuT: (a) c must lie in A, so that $f(c)$ is defined

(b) c may not be a cluster pt of A.

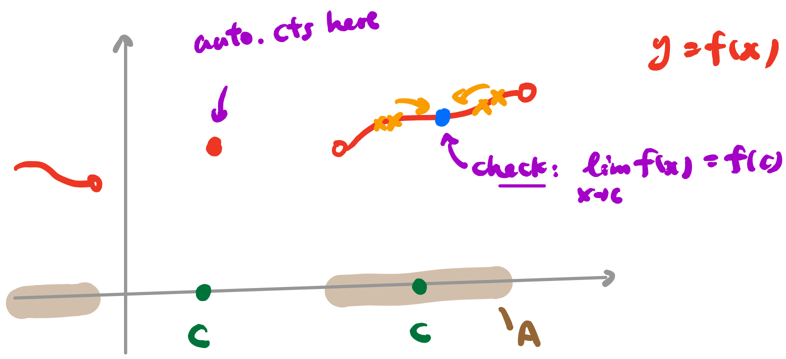
(2) There are 2 cases to consider:

* main Case 1: c is a cluster pt. ($\Rightarrow$ make sense to say $\lim_{x \to c} f(x)$)

" f is cts at c " $\Leftrightarrow$ " $\lim_{x \to c} f(x) = f(c)$ " $(**)$ One can evaluate limit by "substitution".

Case 2: c is NOT a cluster pt.

Then, f is always cts at c. [$\because \exists \delta > 0$ s.t. only $x = c$ satisfy $(*)$]

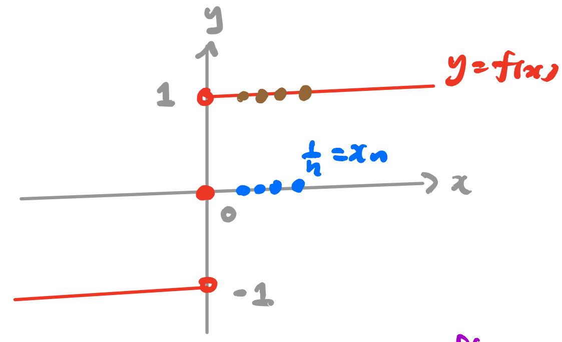


Def$^{\underline{n}}$: f is cts on a subset $B \subseteq A$ iff f is cts at every $c \in B$

Examples of cts fcn: $f(x) = b$, x, x^2, $P(x)$, $\sin x$, $\cos x$, $|x|$

are cts since they satisfy $(**)$

Q: What about fcn which is NOT cts?



Example 1: Consider $f: \mathbb{R} = A \to \mathbb{R}$

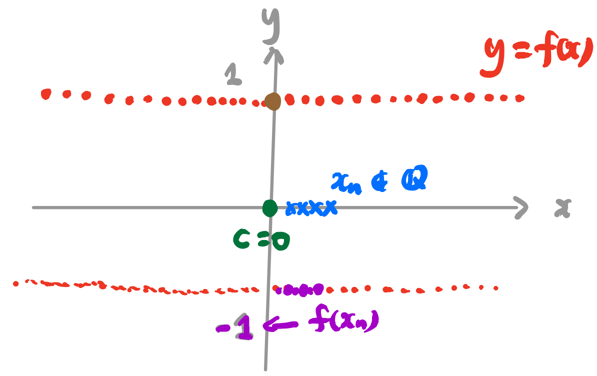
$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Claim: f is cts everywhere except at $x = 0$.

Check: $\lim_{x \to 0} f(x) \neq f(0)$ Since $(x_n) = (\frac{1}{n}) \to 0$, But $(f(x_n)) = (1) \to 1 \neq f(0)$

$\square$

Q: Can a function be discts everywhere?



Example 2: Consider $f: \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Claim: f is discts at every $c \in \mathbb{R}$.

"Pf": Exercise. Main ingredient: $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$

$\square$